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Wind-forced linear and nonlinear Kelvin waves along an irregular coastline

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Using a normal and tangential co-ordinate approach, a perturbation theory is developed for wind-forced linear and nonlinear Kelvin waves propagating along an irregular coastline. The theory is valid for coastline curvatures which are non-dimensionally small, the curvature being non-dimensionalized with respect to the reciprocal of the boundary-layer trapping scale, i.e. the reciprocal of the radius of deformation. According to linear theory, the main effect of a coastline of small curvature is to cause a phase-speed change in the wave (from $-c$ to $-c(1-\frac{1}{2}\kappa(s))$, where $\kappa(s)$ is the non-dimensional curvature a distance s along the coast from the origin) and to make the offshore Ekman transport change more rapidly along the coast, the latter effect implying a more 'wavelike' ocean or lake response. Two discernible nonlinear effects were found to be an increase (decrease) in the linear-solution longshore gradients in regions of positive (negative) isopycnal displacement and a tendency for increased (decreased) isopyncal displacement at capes (bays).

1. Introduction

Recent investigations of the response of coastal waters to atmospheric forcing on time scales much greater than the reciprocal of the Coriolis parameter have shown that there is a strong connexion between the water motion and wind-forced long waves trapped near the coast (see, for example, Walin 1972*a*, *b*; Bennett 1973; Gill & Schumann 1974; Gill & Clarke 1974; Clarke 1977). These investigations effectively obtained results for a straight coast so it is of some interest to examine the wind-forced water motions when, more realistically, the coastline is irregular. Some aspects of this problem have been examined recently by Allen (1976), who ignored stratification effects and concentrated on the influence of large-scale, small amplitude longshore variations in the shelf topography for a straight coastline. To isolate possible effects of *finite* amplitude irregularities in the coastline, this paper presents linear and nonlinear theories for the simple case when the ocean is of constant depth. In such a case the long waves propagating along the coast are Kelvin waves.

Properties of *unforced* linear Kelvin waves propagating along a non-straight coastline have been examined by Buchwald (1968), Packham & Williams (1968), Pinsent (1972), Miles (1972) and Mysak & Tang (1974) for all frequencies. For the subinertial frequencies of interest here, Buchwald found that Kelvin waves were unattenuated after passing around a right-angled corner, Packham & Williams proved this result for a corner of any angle and Miles extended the work of Packham & Williams by finding the time delays which unforced Kelvin waves experience as they pass around the corner. Mysak & Tang adopted a different approach, statistically analysing the effect of coastline irregularities which have a scale small compared with the boundary-layer trapping scale. The mathematical techniques used in all these analyses are quite complicated and not suitable for the wind-forced irregular-coastline case.

As will be seen, the forced case is conveniently examined by adopting a normal and tangential co-ordinate approach. Such a treatment allows not only wind-forced linear, but also wind-forced nonlinear Kelvin waves to be considered. The normal and tangential co-ordinate approach is quite powerful and could probably be usefully employed in the analysis of other physical situations in which motion is trapped near an irregular boundary.†

After formulating a linear, constant-depth, stratified ocean model in § 2, a normal and tangential co-ordinate approach is described in § 3 and used to obtain a simple solution for the pressure. The solution is valid for small κ, where κ is the curvature of the coastline non-dimensionalized with respect to the reciprocal of the boundary-layer trapping scale, i.e. the reciprocal of the radius of deformation. In § 4 the effect of curvature on the phase speed of the wave and a description of how bends in the coastline can cause the solution to be 'wavelike' are discussed. An analysis of the nonlinear solution derived in the appendix is given in § 5 and then a summary and some concluding remarks are presented in § 6.

2. Formulation of a linear model

The linear equations of motion for a constant-depth, rotating, horizontally stratified ocean can be separated into equations involving horizontal variations only by separation into vertical modes (Taylor 1936). Following Gill and Clarke (1974), the equations for the ith mode take the form

$$\partial \mathbf{u}_i/\partial t + f\mathbf{k}\times\mathbf{u}_i = -\nabla p_i + b_i \boldsymbol{\tau}/\rho_0 H_{\mathrm{mix}} = -\nabla p_i + \mathbf{X}_i, \tag{2.1}$$

$$c_i^{-2}\partial p_i/\partial t + \nabla . \mathbf{u}_i = 0. \tag{2.2}$$

In these equations $\mathbf{u}_i$, ∇, $\mathbf{k}$, f, t, c_i^2, b_i, $\boldsymbol{\tau}$, H_{mix} and p_i refer respectively to the horizontal part of the velocity, the horizontal gradient operator, the unit vector pointing upwards from the ocean surface, the Coriolis parameter, the time, a separation constant, a forcing coefficient, the wind stress, the depth of the surface mixed layer of the ocean and the pressure divided by ρ_0 (a representative density of the fluid). For notational convenience, the subscript i will be ignored in the work which follows. The constant c, which has the dimensions of velocity, is the propagation speed of a linear Kelvin wave travelling along a straight coast.

If lengths are non-dimensionalized with respect to the radius of deformation c/f, the horizontal velocity $\mathbf{u}$ with respect to c, t with respect to f^{-1}, p with respect to c^2 and $\mathbf{X}$ with respect to fc, (2.1) and (2.2) become

$$\mathbf{u}_t + \mathbf{k}\times\mathbf{u} = -\nabla p + \mathbf{X}, \tag{2.3}$$

$$p_t + \nabla . \mathbf{u} = 0. \tag{2.4}$$

† As pointed out by a referee, the normal and tangential co-ordinate approach adopted in this paper is novel but elements of it have been used previously (e.g. see Robinson & Niiler 1967).

The variables $\mathbf{u}$, $\mathbf{X}$, ∇, p and t now represent non-dimensional versions of the original variables.

Application of the operator $\partial/\partial t - \mathbf{k}\times$ to (2.3) results in

$$(\partial^2/\partial t^2 + 1)\,\mathbf{u} = -\nabla p_t + \mathbf{k}\times\nabla p - \mathbf{k}\times\mathbf{X} + \mathbf{X}_t. \tag{2.5}$$

Elimination of $\mathbf{u}$ between this equation and (2.4) then gives the field equation

$$\frac{\partial}{\partial t}\left[\nabla^2 p - \left(\frac{\partial^2}{\partial t^2}+1\right)p\right] = \mathbf{k}.\nabla\times\mathbf{X} + \nabla.\mathbf{X}_t, \tag{2.6}$$

which is subject to the boundary conditions that $p \to 0$ at large distances from the coast and that the velocity normal to the irregular boundary is zero at the boundary.

3. Method of solution

One would expect that coastline irregularities which have a scale which is everywhere small compared with the boundary-layer scale, the radius of deformation, would have little effect on the solution. Evidence supporting this view has been provided by Mysak & Tang (1974), who developed a linear, unforced, statistical perturbation theory for coastlines having such irregularities. They found that the correction to the solution due to the irregularities was negligible, being of order ϵ^2, where

$$\epsilon = \frac{\text{typical scale of coastline irregularities}}{\text{radius of deformation}}.$$

Therefore, with little error, the model coastline can be defined as a 'smooth' boundary such that the real coastline departs from the smooth boundary only by perturbations which are small compared with the radius of deformation.

Of all possible smooth coastlines, the aim here will be to develop a method of solution which is valid for smooth coastlines which are 'slowly varying' in the sense that the radius of curvature is always large compared with the radius of deformation. In non-dimensional units, this condition is simply that the radius of curvature A is always large. It will be found that at zero order the solution is the same as for a straight coast. The purpose of the analysis to be presented is to calculate to order A^{-1} the modification by the curvature.

Consider normal and tangential co-ordinates (n, s) to the coastline (figure 1). n is defined as the non-dimensional distance seawards from the coast and s as the non-dimensional distance along the coast from the origin. Since n denotes non-dimensional distance from the coast, the equation of the coast is $n = 0$ and the equation of a vertical surface a distance of one radius of deformation from the coast is $n = 1$. According to this definition, some points in the fluid could be associated with more than one (n, s) pair; this is avoided by defining n at those points to be the smaller value (see figure 1).

The curvature $\kappa(s)$ ($|\kappa| = 1/A$) is defined to be positive (negative) when the coastline is convex seawards (landwards). By analogy with cylindrical polar co-ordinates, it might be expected that the metrics d_1 and d_2 corresponding to the general orthogonal co-ordinates (n, s) would be given by

$$d_1 = 1, \quad d_2 = 1 + n\kappa(s).$$

This result is in fact valid and can be checked using simple vector differential geometry (see Clarke 1976).

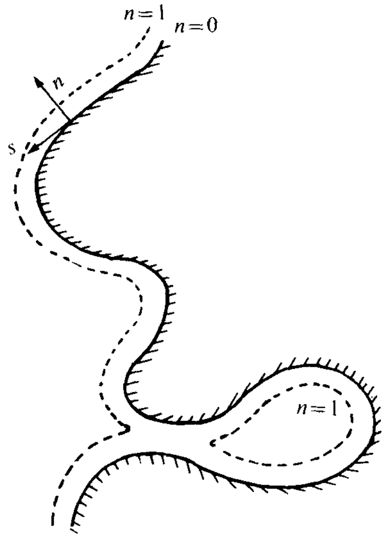

FIGURE 1. Co-ordinates normal and tangential to the coastline. The contours do not connect between the bay and the open sea in the bottom half of the figure because n is defined to be the shortest distance from the coast.

In terms of normal and tangential co-ordinates, (2.6) can thus be written as

$$\frac{\partial}{\partial t}\left[\frac{1}{1+n\kappa}\frac{\partial}{\partial s}\left(\frac{1}{1+n\kappa}\frac{\partial p}{\partial s}\right)+\kappa\frac{\partial p}{\partial n}\Big/(1+n\kappa)+\frac{\partial^2 p}{\partial n^2}-\left(\frac{\partial^2}{\partial t^2}+1\right)p\right] = \mathbf{k}.\nabla\times\mathbf{X}+\nabla.\mathbf{X}_t. \tag{3.1}$$

Since the coastline has a longshore scale of order A and the wind generally has a scale at least as large as the typical values of A of interest, the longshore scale of the motion is of order A or larger. Therefore, provided $1+n\kappa$ is of order 1 or larger, (3.1) can be written as

$$\frac{\partial}{\partial t}\left[O(\kappa^2)+\kappa\frac{\partial p}{\partial n}\Big/(1+n\kappa)+\frac{\partial^2 p}{\partial n^2}-\left(\frac{\partial^2}{\partial t^2}+1\right)p\right] = \mathbf{k}.\nabla\times\mathbf{X}+\nabla.\mathbf{X}_t. \tag{3.2}$$

The restriction on $1+n\kappa$ means that the solution is valid everywhere except where the curvature of the coast is negative and n is approximately equal to A. Since the boundary-layer scale is the radius of deformation, all the motion of interest is in the region where the solution is valid.

The right-hand side of (3.2) suggests that the motion is driven by the wind-stress curl or divergence, but this is not so: the motion is principally driven by the Ekman flux perpendicular to the coast [see (3.10)]. It can be verified *a posteriori* that the error introduced by neglecting the right-hand side of (3.2) is of order L^{-1}, where L is the ratio of the wind stress to the radius of deformation. L is generally very large. T, the time scale non-dimensionalized with respect to f^{-1}, is also large (only time scales much greater than f^{-1} will be considered), so when (3.2) is written as

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 p}{\partial n^2}+\kappa\frac{\partial p}{\partial n}\Big/(1+n\kappa)-p\right] = 0+O\left(\max\left\{\frac{\kappa^2}{T},\frac{1}{TL},\frac{1}{T^3}\right\}\right) \tag{3.3}$$

the right-hand side is very small compared with the left-hand side. For simplicity, the right-hand side will be taken to be $\leqslant O(\kappa^3)$. If the motion begins from a state of rest then (3.3) can be integrated with respect to time to give

$$\frac{\partial^2 p}{\partial n^2}+\kappa\frac{\partial p}{\partial n}\Big/(1+n\kappa)-p = 0+O(\kappa^2). \tag{3.4}$$

Since it is required that $p \to 0$ as $n \to \infty$,† the solution of this equation when the curvature is positive is

$$p = \phi(s,t)\,K_0\left(\frac{1+n\kappa}{\kappa}\right)\Big/K_0\left(\frac{1}{\kappa}\right) = \phi(s,t)\left(\frac{1}{1+n\kappa}\right)^{\frac{1}{2}}e^{-n}\left\{1-\frac{\kappa}{8(1+n\kappa)}+\frac{\kappa}{8}+O(\kappa^2)\right\} \tag{3.5}$$

and for negative curvature is

$$p = \phi(s,t)\,I_0\left(\frac{1+n\kappa}{\kappa}\right)\Big/I_0\left(\frac{1}{\kappa}\right) = \phi(s,t)\left(\frac{1}{1+n\kappa}\right)^{\frac{1}{2}}e^{-n}\left\{1-\frac{\kappa}{8(1+n\kappa)}+\frac{\kappa}{8}+O(\kappa^2)\right\}. \tag{3.6}$$

In these equations I_0 and K_0 are modified Bessel functions and $\phi(s, t)$ is an arbitrary function. When the curvature is small and the frequency squared is small compared with f^2, this solution generalizes the circular-island solution of Longuet-Higgins (1969) and the circular-lake solution of Csanady (1972). A circular island corresponds to the case when κ is positive and constant, a circular lake to κ constant and negative.

The motion is trapped within roughly one radius of deformation of the coast so in this region of interest n is small compared with A. Applying this approximation to (3.4) gives

$$\partial^2 p/\partial n^2+\kappa\,\partial p/\partial n-p = 0 \tag{3.7}$$

and so

$$p = \phi(s,t)\,e^{-n}(1-\tfrac{1}{2}n\kappa)+O(\kappa^2). \tag{3.8}$$

Both (3.5) and (3.6) reduce to (3.8) for n small compared with A.

The remaining boundary condition to be satisfied is that the velocity component normal to the coast is zero at the coast. The velocity component normal to the coast can be obtained by taking the scalar product of the unit normal to the coast with both sides of (2.5). To within an error of order κ^2 or smaller the boundary condition can be written as

$$\partial p/\partial s+\partial^2 p/\partial n\,\partial t = \hat{\mathbf{e}}_n.\mathbf{X}_t+\hat{\mathbf{e}}_s.\mathbf{X} \quad \text{on} \quad n = 0, \tag{3.9}$$

where the unit vectors $\hat{\mathbf{e}}_n$ and $\hat{\mathbf{e}}_s$ are in the direction of increasing n and s respectively. Using (3.8) and (3.9) gives the following equation for ϕ:

$$\frac{1}{(-1+\frac{1}{2}\kappa)}\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial s} = \hat{\mathbf{e}}_n.\mathbf{X}_t+\hat{\mathbf{e}}_s.\mathbf{X}. \tag{3.10}$$

A convenient way to solve (3.10) is to integrate along the characteristic defined by

$$ds/dt = -1+\tfrac{1}{2}\kappa. \tag{3.11}$$

Along this characteristic (3.10) becomes

$$d\phi/dt = (\hat{\mathbf{e}}_n.\mathbf{X}_t+\hat{\mathbf{e}}_s.\mathbf{X})\,(-1+\tfrac{1}{2}\kappa), \tag{3.12}$$

so that (3.10) has been converted into a form which can readily be integrated.

† There are coastlines for which n as defined cannot tend to infinity (see figure 1). Under the mild restriction that n can always increase to a number greater than 1, the condition $p \to 0$ as $n \to \infty$ is a reasonable approximation.

4. Analysis of the solution

For a straight coast ($\kappa \equiv 0$), (3.8), (3.11) and (3.12) reduce to the Kelvin-wave equations discussed in Gill & Clarke (1974). When the coastline is not straight, the decay of the Kelvin wave with distance from the shore, the speed of the Kelvin wave and the wind forcing term are all changed. These changes are discussed below.

When there is no wind the solution of (3.10) represents a free wave propagating at a non-dimensional speed given by (3.11). In dimensional units the change in speed is from $-c$ (straight coast) to $(-1+\frac{1}{2}\kappa)\,c$; the Kelvin wave thus speeds up when travelling along a bay ($\kappa < 0$) and slows down at capes ($\kappa > 0$). This result agrees with the results of a ray-theory approach (Clarke, unpublished) and equation (1.5) of Miles (1972) (for the same angular change in coastline direction).

As well as the phase-speed change, the decay of the Kelvin wave with distance from the coast is changed when the coast is curved. However, the extra linear variation with n is unimportant compared with the exponential decay and it can still be said that the solution has a coastal boundary-layer width scale equal to the radius of deformation.

In (3.12), since $\partial/\partial t \ll 1$, $|\hat{\mathbf{e}}_s.\mathbf{X}| \gg |\hat{\mathbf{e}}_n.\mathbf{X}_t|$ or both terms are insignificant. Thus, although the expression for the wind forcing term has changed, the physics expressed by (3.12) are the same as those expressed by the first-order wave equation in the straight-coast case: the rate of change of ϕ for an observer moving with the wave is proportional to the longshore component of the wind and hence the Ekman transport perpendicular to the shore. This implies that the amplitude ϕ at a given coastal position depends on how much it has been increased or decreased by Ekman transport as the Kelvin wave has propagated along the coast to the coastal position under consideration.

Note particularly that, since the wave is generated by the wind, its amplitude need not be sinusoidal in shape and will in fact depend on the (s, t) structure of $\hat{\mathbf{e}}_s.\mathbf{X}$. When $\hat{\mathbf{e}}_s.\mathbf{X}$ is independent of the longshore co-ordinate s, then under the assumption of an initial state of rest the solution of (3.10) is independent of s and no waves can be observed. This last result implies that it should be more difficult to observe wind-generated long-wave propagation along a straight coast (where $\hat{\mathbf{e}}_s.\mathbf{X}$ is weakly dependent on s) than for (say) a large lake (where $\hat{\mathbf{e}}_s.\mathbf{X}$ is much more strongly dependent on s owing to bends in the 'coast'). Observational evidence (e.g. from the Oregon coast and Lake Ontario) tends to support this view (see Clarke 1977).

To summarize, according to *linear* theory, the main effect of a slowly varying coastline is to cause a change in the phase speed of the wave and to make the offshore Ekman transport change more rapidly along the coast, the latter effect implying a more 'wavelike' ocean or lake response.

5. Nonlinear effects

When the ocean, a sea or a large lake is influenced by the action of a strong wind blowing parallel to the shore, it is usually the case that near the shore the pycnocline is displaced a vertical distance comparable to the pycnocline depth in the water at rest,

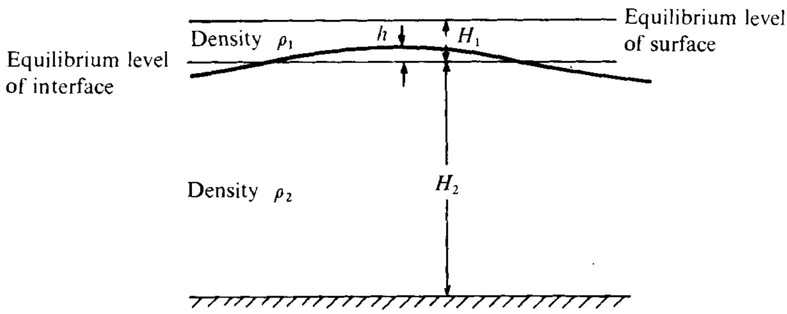

FIGURE 2. A diagram illustrating the definition of H_1, H_2 and h.

and this suggests that nonlinear effects could be important. The analysis of the wind-forced *nonlinear* motion of arbitrarily stratified water along an irregular coastal boundary is difficult. However, some aspects of the nonlinear motion can be analysed analytically if a normal and tangential co-ordinate approach is adopted and several simplifying assumptions are made.

Consider a constant-depth ocean or lake with a smooth,† slowly varying (in the sense discussed earlier) coastline. Let the stratification be approximated by two layers of fluid of constant density, the surface layer being thin compared with the bottom layer (see figure 2). For such a model the nonlinear reduced-gravity equations apply for the single baroclinic mode. If it is assumed, as in the linear case, that the time scales to be considered are long compared with f^{-1}, that the wind has a scale at least as great as A^2 and that the water is initially at rest, then the potential-vorticity equation can be used to derive a perturbation solution for small κ. Details are given in the appendix. The solution can be written [see (A 8), (A 9), and (A 11)] as

$$h = Ie^{-n} + \kappa(Je^{-n} + \tfrac{2}{3}I^2e^{-2n} - \tfrac{1}{2}nIe^{-n}) + O(\kappa^2), \tag{5.1}$$

where h refers to the pycnocline displacement non-dimensionalized with respect to the depth H_1 of the upper layer. I and J are determined from the integration of [see (A 10), (A 12) and (A 13)]

$$dI/dt = \hat{\mathbf{e}}_s.\mathbf{X} \tag{5.2}$$

and

$$\frac{d}{dt}\left[\kappa J \exp\left(\int_0^t I_s\,dt_*\right)\right] = X_I \exp\left(\int_0^t I_s\,dt_*\right) \tag{5.3}$$

along the characteristic

$$ds/dt = -1 + I. \tag{5.4}$$

Bennett (1973) obtained this solution for a straight coast ($\kappa \equiv 0$); in his analysis rectangular Cartesian axes x and y replaced the n and s axes and $\mathbf{j}$, the constant unit vector in the y direction, replaced $\hat{\mathbf{e}}_s$.

Discussion of the zeroth-order solution

To discuss the nonlinear effects, consider first the zeroth-order solution. The physics expressed by (5.2) and (5.4) are essentially the same as those for the linear case except

† As pointed out by a referee, in the linear case the real coastline can be 'smoothed' without introducing a serious error, but in the nonlinear case perturbations to a smooth coastline could well introduce an interesting nonlinear effect. Such an effect will not be investigated here.

that the phase speed of the wave now depends on its amplitude, decreasing as the pycnocline depth is decreased. When the pycnocline actually reaches the surface at the coast,† the nonlinear Kelvin-wave phase speed is in theory zero. In practice, however, the mixing action of the wind causes the pycnocline to become diffuse as it nears the surface so one might in fact expect 'noisy' propagation at a reduced speed.

It is interesting that, for a straight coastline and a spatially homogeneous wind, the linear and nonlinear solutions are the same: both are uniform along the coast and are given by

$$h = \left(\int_0^t (\hat{\mathbf{e}}_s . \mathbf{X})\, dt_*\right) e^{-n}. \tag{5.5}$$

When $\hat{\mathbf{e}}_s . \mathbf{X}$ depends on s, however, (5.5) no longer applies and linear and nonlinear wave properties become important. Consider, for example, the simple case when $\hat{\mathbf{e}}_s . \mathbf{X}$ is positive everywhere except for an isolated section of coast where it is zero. For convenience, label the positive sections OA and BC, the zero section AB and let the direction of Kelvin-wave propagation be $CBAO$. A simple characteristics analysis (details are given in Clarke 1976) shows that, from an initial state of rest, the pycnocline height increases not only in the regions OA and BC, where $\hat{\mathbf{e}}_s . \mathbf{X}$ is positive, but also, owing to Kelvin-wave propagation, in the zero region AB. Note however that, since the wave arriving at A at any time t has had to traverse a section of coast BA where $\hat{\mathbf{e}}_s . \mathbf{X}$ is zero, it is in general forced less than the one arriving at B at time t and consequently there is a drop in pycnocline height from B to A. This drop is greater for the nonlinear case because, when the pycnocline displacement is positive, nonlinear Kelvin waves have a smaller phase speed than the linear waves and consequently spend a longer time in the section BA of no forcing. If OA and BC are regions where $\hat{\mathbf{e}}_s . \mathbf{X}$ is negative instead of positive the reverse results apply: *downwelling* (i.e. downward pycnocline displacement) develops along OA and BC and, owing to the *greater* nonlinear propagation speed, the change in pycnocline height from B to A is *smaller* in the nonlinear than the linear case.

Discussion of the first-order solution

The *first*-order solution [see (5.1)–(5.4) and (A 13)] is in general complicated. However, since κJ is altered only when κ or κ_s is non-zero for a large class of coastlines (e.g. those that have $\kappa = 0$ along most of the coastline), $|J|$ is small compared with $|I|$. When this condition is satisfied, the pycnocline height close to the coast can be approximated by the simple equation

$$h = I\, e^{-n} + \tfrac{2}{3}\kappa I^2 e^{-2n}, \tag{5.6}$$

where I is determined from (5.2). This result suggests that there is a tendency for isopycnal displacement to be increased in regions of positive curvature ('capes') and decreased in regions of negative curvature ('bays'). It is interesting to speculate whether this result is still qualitatively true for moderate or large curvature. The results of the nonlinear two-layer numerical model of Hurlburt (1974) suggest that for $|\kappa| \approx 2$ there is indeed a tendency for isopycnal displacement to be increased at 'capes' and decreased at 'bays', albeit that the increase and decrease are considerably less than those which would be obtained by putting $\kappa = \pm 2$ in (5.6). Although it is

† It should be noted that the theory breaks down when the pycnocline reaches the surface and the wind remains in a direction which would normally lead to increased pycnocline height.

encouraging to see that the perturbation solution appears to model qualitatively a solution when κ is of order 1, it should be mentioned that there is some doubt as to the validity of the numerical results where $|\kappa| = 2$ because the space scales of the solution near those sections of the coast are only a few times larger than the grid size.

6. Summary and concluding remarks

In addition to demonstrating a way of analysing trapped motions along an irregular boundary, this paper has attempted to isolate and examine some aspects of the influence of coastline irregularities on the distribution of upwelling and currents along a coastline. Using simple models, results were obtained for the wind-forced motion of stratified water of constant depth near a coastline which varies on a scale much larger than the radius of deformation. For the linear model it was found that the solution can be described by a wind-forced Kelvin wave, the speed of which depends on the non-dimensional curvature $\kappa(s)$ of the coastline. The character of the water response is dependent on the coastline type. For long, straight or nearly straight coastlines the offshore Ekman transport is weakly dependent on the longshore coordinate s, and consequently, even though the wave mechanism operates, waves are not easily observed. For a bent coastline, however, the offshore Ekman transport is more strongly dependent on s and propagating waves can in fact be observed. A review of the observational evidence for such forced propagating waves (and a discussion of the effect of curvature on the propagating wave in Lake Ontario) has been given in Clarke (1977).

In the nonlinear case, the phase speed of the Kelvin wave depends on its amplitude, so longshore gradients of the nonlinear solution are greater in an upwelling region and less in a downwelling region than in the linear case. Another identifiable nonlinear effect (at least for coastlines of small curvature and the two-layer stratification considered) is the tendency for upwelling to be increased at 'capes' ($\kappa > 0$) and to be reduced at 'bays' ($\kappa < 0$). There is some computational evidence to suggest that this effect may persist when κ is of order 1, but to the author's knowledge no ocean or lake measurements have been made with which this effect could be adequately tested.

In conclusion, it should be stressed that other factors besides the coastline configuration can strongly influence the longshore variation of upwelling and currents. The most notable of these, longshore variations in bottom topography, has been discussed numerically by Peffley & O'Brien (1976) and observationally by Shaffer (1974) and Copenhagen (1953). The available evidence indicates that longshore topographic variations, usually in the form of canyons running at right angles to the coast, can strongly influence the longshore upwelling distribution on scales of the order of the shelf width or radius of deformation. Such scales are generally significantly smaller than the coastline scales discussed in this paper, so it seems that, while coastline variations may have the stronger influence on scales large compared with the radius of deformation, on smaller scales longshore topographic effects may dominate. However, such a conclusion ignores the possibly important nonlinear small-scale influence of pronounced capes of large curvature. Further study is clearly necessary to resolve these and other questions, but such a study is beyond the scope of this paper.

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Appendix. Nonlinear theory

Under the assumptions discussed in §5, potential vorticity is conserved for the two-layer nonlinear model. Thus

$$\frac{D}{Dt}\left(\frac{\zeta+1}{-h+1}\right) = \left(\frac{\partial}{\partial t}+\mathbf{u}.\nabla\right)\left(\frac{\zeta+1}{-h+1}\right) = 0. \tag{A 1}$$

Here h denotes the pycnocline displacement non-dimensionalized with respect to the depth H_1 of the upper layer, ζ denotes the vertical component of relative vorticity non-dimensionalized with respect to f, u denotes the upper-layer velocity non-dimensionalized with respect to c† and the other non-dimensional symbols have the same meaning as in §2. As in the linear case, the fluid starts from rest, so initially the potential vorticity $(\zeta+1)/(-h+1)$ has the value 1 everywhere. Equation (A 1) indicates that if the potential vorticity initially has the value 1 then it must have this value for all time. Hence the field equation for the motion is

$$\zeta + h = 0. \tag{A 2}$$

To obtain the field equation in terms of h alone, ζ must be calculated in terms of h by taking the horizontal curl of the nonlinear analogue of (2.5). One has

$$(1+O(\kappa^2))\,\mathbf{u} = \frac{D}{Dt}(\nabla h) - \mathbf{k}\times\nabla h - \mathbf{k}\times\mathbf{X} + \frac{D\mathbf{X}}{Dt} \tag{A 3}$$

and

$$\zeta\mathbf{k} = \nabla\times\mathbf{u} = \nabla\times(\mathbf{u}.\nabla(\nabla h)) - (\nabla^2 h)\,\mathbf{k} + O(\kappa^2). \tag{A 4}$$

By repeated substitution for $\mathbf{u}$ and use of the results

$$\frac{\partial\hat{\mathbf{e}}_n}{\partial n} = \frac{\partial\hat{\mathbf{e}}_s}{\partial n} = 0, \quad \frac{\partial}{\partial s}\hat{\mathbf{e}}_n = \kappa\hat{\mathbf{e}}_s, \quad \frac{\partial\hat{\mathbf{e}}_s}{\partial s} = -\kappa\hat{\mathbf{e}}_n,$$

which follow from an appendix to Batchelor (1967), it can be shown that

$$\nabla\times(\mathbf{u}.\nabla(\nabla h)) = -2\kappa h_n h_{nn}\,\mathbf{k} + O(\kappa^2), \tag{A 5}$$

where the subscript n denotes $\partial/\partial n$. Equations (A 2), (A 4) and (A 5) together imply that the field equation is

$$h_{nn} + \kappa h_n + 2\kappa h_n h_{nn} - h = 0 \tag{A 6}$$

to order κ.

This equation is to be solved subject to the condition that $h\to 0$ as $n\to\infty$ and the

† The phase speed c of long internal waves in the linearized version of the reduced-gravity equations is (Stokes 1847)

$$c = \left[\frac{g\Delta\rho}{\rho_0}\frac{H_1 H_2}{(H_1+H_2)}\right]^{\frac{1}{2}} \approx \left(\frac{g\Delta\rho H_1}{\rho_0}\right)^{\frac{1}{2}},$$

where $\Delta\rho$ is the density difference between the upper and lower layers, H_2 is the depth of the lower layer and g is the acceleration due to gravity.

condition that the velocity component normal to the coast is zero at the coast. From the $O(1)$ balance in (3.10) and the result $\partial/\partial s = O(\kappa)$, the coastal boundary condition can be written in terms of h as

$$-h_{nt}+h_{ns}h_n^2\kappa+h_{ns}h_n-\kappa h_s h_n-h_s = \hat{\mathbf{e}}_n.\mathbf{X}_t+\hat{\mathbf{e}}_s.\mathbf{X}+\kappa h_n(\hat{\mathbf{e}}_s.\mathbf{X})-h_{ns}(\hat{\mathbf{e}}_n.\mathbf{X})+O(\kappa^3) \quad \text{on} \quad n=0. \qquad (\text{A }7)$$

The nonlinear problem defined by (A 6), (A 7) and the boundary condition at infinity can be solved for small κ using a perturbation approach. Write

$$h = h^{(0)}+\kappa h^{(1)}+\dots. \qquad (\text{A }8)$$

The zeroth-order problem is

$$h^{(0)}_{nn}-h^{(0)} = 0,$$

$$h^{(0)}\to 0 \quad \text{as} \quad n\to\infty$$

and

$$-h^{(0)}_{nt}+h^{(0)}_{ns}h^{(0)}_n-h^{(0)}_s = \hat{\mathbf{e}}_s.\mathbf{X} \quad \text{on} \quad n=0.$$

The solution is

$$h^{(0)} = Ie^{-n}, \qquad (\text{A }9)$$

where I satisfies

$$I_t+I_s(-1+I) = \hat{\mathbf{e}}_s.\mathbf{X}. \qquad (\text{A }10)$$

The first-order problem is

$$h^{(1)}_{nn}-h^{(1)} = Ie^{-n}+2I^2e^{-2n},$$

$$h^{(1)}\to 0 \quad \text{as} \quad n\to\infty$$

and

$$-\kappa h^{(1)}_{nt}+(\kappa h^{(1)}_n)_s h^{(0)}_n+\kappa h^{(1)}_n h^{(0)}_{ns}-(\kappa h^{(1)})_s = \hat{\mathbf{e}}_n.\mathbf{X}_t+I_s(\hat{\mathbf{e}}_n.\mathbf{X})-\kappa(\hat{\mathbf{e}}_s.\mathbf{X})I+I^2I_s\kappa-\kappa II_s \quad \text{on} \quad n=0.$$

The solution is

$$h^{(1)} = J(s,t)e^{-n}+\tfrac{2}{3}I^2e^{-2n}-\tfrac{1}{2}nIe^{-n}, \qquad (\text{A }11)$$

where κJ satisfies

$$(\kappa J)_t+(\kappa J)_s(I-1)+(\kappa J)I_s = X_I \qquad (\text{A }12)$$

and

$$X_I = \hat{\mathbf{e}}_n.\mathbf{X}_t+I_t[-\tfrac{8}{3}I\kappa-\tfrac{1}{2}\kappa]+\kappa_s[-\tfrac{4}{3}I^3+\tfrac{1}{6}I^2]+\kappa[-3I^2I_s-\tfrac{2}{3}II_s] +I_s(\hat{\mathbf{e}}_n.\mathbf{X})-\kappa I(\hat{\mathbf{e}}_s.\mathbf{X}). \qquad (\text{A }13)$$

Note that equations (A 10) and (A 12) for I and J can be simplified by integrating along the characteristic

$$ds/dt = -1+I \qquad (\text{A }14)$$

(see § 5).

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